

Localization, Dirac fermions and Onsager universality

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Abstract. Disordered systems exhibiting exponential localization are mapped to anisotropic spin chains with localization length being related to the anisotropy of the spin model. This relates localization phenomenon in fermions to the rotational symmetry breaking in the critical spin chains. One of the intriguing consequence is that the statement of Onsager universality in spin chains implies universality of the localized fermions where the fluctuations in localized wave functions are universal. We further show that the fluctuations about localized nonrelativistic fermions describe relativistic fermions. This provides a new approach to understand the absence of localization in disordered Dirac fermions. We investigate how disorder affects well known universality of the spin chains by examining the multifractal exponents. Finally, we examine the effects of correlations on the localization characteristics of relativistic fermions.

PACS. 75.10.Jm Quantized spin models – 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.) – 68.35.Rh Phase transitions and critical phenomena

Two-dimensional Ising model is one of the few examples of exactly solvable many body systems [1]. The model exhibits a phase transition at finite temperature characterized by universal exponents defining a universality class, the Onsager universality which describes the phase transitions for anisotropic XY models. Interestingly, the Onsager universality also describes a quantum phase transition at zero temperature, driven by quantum fluctuations, of one-dimensional quantum anisotropic XY spin chains in a transverse field [2]. These quantum models belong to a small family of integrable Hamiltonians that have attracted both theoreticians as well as experimentalists. Heart of integrability of these many body quantum spin problem is a mapping between spin and fermions which relates interacting XY spin chain with $O(1)$ symmetry to the fermion Hamiltonian that are quadratic in fermions. The spin-fermion correspondence has proven to be extremely important also for the case of disordered magnetic field as the quadratic fermion Hamiltonian can be numerically diagonalized with extreme precision [4]. Recent studies have shown that the a large variety of disordered quantum chains with $O(1)$ spin symmetry are still described by the Onsager universality [3,4].

In this paper, we show a new type of spin-fermion mapping where disordered fermions exhibiting exponential localization are shown to be related to anisotropic spin chain in a disordered magnetic field at the onset to long range order (LRO). This correspondence, valid exclusively for exponentially localized systems, provides new insight into various issues relevant to disordered fermion as well as

spin models. In particular we exploit well established universality hypothesis of spin systems to make important statements about fermion problems. Firstly, the root of recently observed universality in localized fermions [9,10] is traced to the Onsager universality of the spin systems. Another interesting result is a correspondence between the relativistic and the nonrelativistic fermions in the presence of disorder. It is shown that the relativistic fermions can be viewed as the fluctuations in the exponentially localized solutions of the nonrelativistic fermions. This provides a new approach for understanding the absence of localization in disordered Dirac fermions which has been the subject of various recent studies [11]. We argue that the long range magnetic correlations provide mechanism for delocalization of relativistic fermions thus obtaining a deeper understanding of the absence of localization. In addition to obtaining intuitively appealing picture of some surprising results of disordered fermions, we also obtain a generalization of universality statement of the critical exponents for disordered spin models. Finally, we examine how the correlations affect the localization characteristics and show the possibility of delocalization of relativistic fermions analogous to the corresponding nonrelativistic case.

Although the setting we describe is quite general in the context of disordered systems, for concreteness we will consider quasiperiodic disorder where the lattice problem for nonrelativistic fermion is described by the Harper equation [12],

$$\psi_{n-1} + \psi_{n+1} + 2\lambda V_n \psi_n = E\psi_n. \quad (1)$$

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Here $V_n = \cos(\theta_n)$ where $\theta_n = (2\pi\sigma_g n + \phi)$. The σ_g is an irrational number describing competing length scales in the problem and ϕ is a constant phase factor. Harper equation, in one-band approximation describes Bloch electrons in a magnetic field. This problem frequently arises in many different physical contexts, every time emerging with a new face to describe another physical application. The problem is solvable by Bethe-ansatz [5]: there is an algebraic Bethe-ansatz equation for the spectrum. It was shown that at some special points in the spectrum, *e.g.* at the mid-band points, the Hamiltonian in certain gauge can be written as a linear combination of generators of a quantum group called $U_q(sl_2)$. It also describes some properties of the integer quantum Hall effect: it showed that Kubo-Greenwood formula for the conductance of any filled isolated band is an expression for a topological invariant, and is an integer multiple of e^2/h . It was further shown by Avron *et al.* [6] that it is a topological invariant that defines the first Chern class of the mapping of the Brillouin zone (a two-dimensional torus) onto a complex projective space of the wave functions.

In contrast to usual Anderson problem describing localized particle in a random potential, the Harper equation exhibits localization-delocalization transition provided σ_g is an irrational number with good Diophantine properties (*i.e.* badly approximated by rational numbers). This transition at $\lambda = 1$, the self-dual point, where the model and its Fourier transform become identical, has been characterized with singular continuous states and spectra. Richness and complexity of the critical point describing localization transition has been studied in great detail by various renormalization group approaches [7, 8]. Recently, it was shown that multifractal characteristics continue to exist beyond the critical point [9] throughout the localized phase. This hidden complexity of the localized phase is brought to light after one factors out the exponentially decaying envelope. The localized wave functions with inverse localization length $\xi^{-1} = \log(\lambda)$ is rewritten as [9],

$$\psi_n = e^{-\gamma|n|}\eta_n \quad (2)$$

where $\gamma = \xi^{-1}$. The tight binding model (tbm) describing the fluctuations η_n in the exponentially decaying envelope is given by the following pair of equations,

$$\begin{aligned} e^{-\gamma}\eta_{n+1}^r + e^{\gamma}\eta_{n-1}^r + 2\lambda V_n \eta_n^r &= E\eta_n^r \\ e^{\gamma}\eta_{n+1}^l + e^{-\gamma}\eta_{n-1}^l + 2\lambda V_n \eta_n^l &= E\eta_n^l. \end{aligned} \quad (3)$$

Here η^r and η^l respectively describe the fluctuations to the right and to the left of the localization center. An exact renormalization scheme [9] showed that these fluctuations exhibit universal features (see Fig. 1) described by the strong coupling fixed point. Hence, localized phase is characterized by universal fractal characteristics described by $\lambda \rightarrow \infty$ limit of the equation. These results were further confirmed by rigorous mathematical analysis [10]. Hence the strength of the disorder which determines the localization length can be factored out making universal aspects transparent.

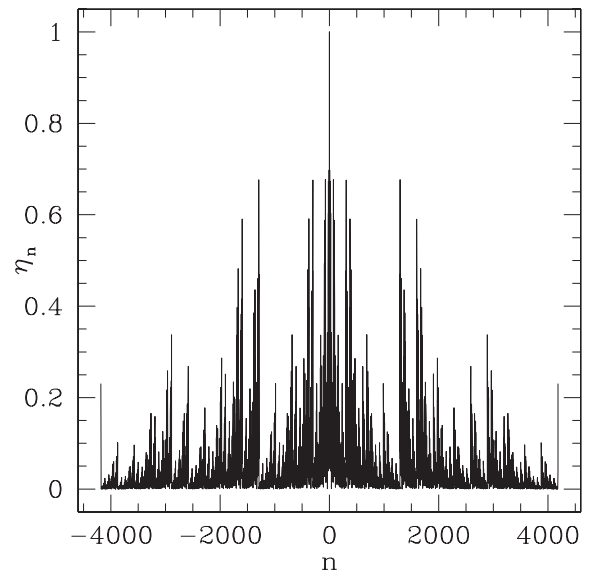


Fig. 1. Absolute value of the fluctuations for Harper equation for $E = 0$ states with $\phi = 0.25$ and $\sigma_g = (\sqrt{5} - 1)/2$. At the Fibonacci sites, we see period-6 behavior (period-3 in absolute value): $|\eta_{F_m}| = |\eta_{F_{m+3}}|$ which is independent of λ .

A new revelation that provides intuitive understanding of the strong coupling fixed point of Harper is obtained by relating the fluctuations described by (3) to the anisotropic spin chain at the onset to LRO. It turns out that the equations (3) for $E = 0$ describe quasiparticle excitations of a *critical* anisotropic XY spin-1/2 chain in a transverse magnetic field, given by the following spin Hamiltonian

$$H = - \sum [e^{-\gamma}\sigma_n^x \sigma_{n+1}^x + e^{\gamma}\sigma_n^y \sigma_{n+1}^y + 2\lambda V_n \sigma_n^z]. \quad (4)$$

The σ_k , $k = x, y$ are the Pauli matrices. The $e^{-\gamma}$ and e^{γ} respectively describe the exchange interactions along the x and the y directions in spin space. Using Jordan-Wigner transformation, the interacting spin problem can be mapped to non-interacting spinless fermion problem [2] where fermions are the quasiparticle excitations of the spin chain obeying the following coupled equations,

$$\begin{aligned} e^{-\gamma}\eta_{i+1}^1 + e^{\gamma}\eta_{i-1}^1 + 2\lambda V_i \eta_i^1 &= E\eta_i^2 \\ e^{\gamma}\eta_{i+1}^2 + e^{-\gamma}\eta_{i-1}^2 + 2\lambda V_i \eta_i^2 &= E\eta_i^1. \end{aligned} \quad (5)$$

At the onset to LRO, the excitation spectrum becomes gap-less and hence for the $E = 0$, the massless mode, the above two equations are degenerate coinciding with the equation (3). Therefore, the massless excitations of the critical anisotropic spin chain describe the fluctuations in the exponentially localized excitations of the isotropic chain where the anisotropy parameter is related to the localization length.

Therefore, the localization length like the anisotropy parameter is an irrelevant variable and hence the statement of strong coupling universality of the Harper equation is synonymous with the statement that anisotropic spin chain is in the universality class of the Onsager universality. In short, we establish an equivalence between

Ising fixed point of the anisotropic spin chain and the strong coupling fixed point of the Harper equation and thus provide a simple interpretation of the strong coupling universality of the Harper equation. It should be noted that the spin-fermion mapping provides a new method to determine the localization length of the tbms in the presence of disorder as the equation for the critical point determines the localization length.

It should be noted that the Fluctuations in the localized wave functions of the fermions are universal for all values of energy [9] and quasiparticle excitations of the spin chains are also believed to be universal for all energies. However, we have been able to establish this equivalence only for $E = 0$ state, the massless mode responsible for LRO. The question whether the universality of all states in equations (3) and (5) are related is an interesting open question.

We next show that the fluctuation η_n obey Dirac equation for zero energy states. In the long wave length limit, the equations(3) reduce to the Dirac equation. We replace n by x and write $\eta_{n\pm 1} = e^{\pm ip}\eta(x)$, where p is the momentum canonically conjugate to x . The equation (4) for the fluctuations for $E = 0$ state can be described by the following non-Hermitian Hamiltonian H_{fluc} and its adjoint, $H_{fluc} = e^{-\gamma}e^{ip} + e^{\gamma}e^{-ip} + 2\lambda V(x)$. In the limit ($p \rightarrow 0$), the system for $E = 0$ reduces to the Dirac equation,

$$[g\sigma_x p - i(2\lambda V(x) + 2 \cosh(\gamma))\sigma_y]\eta(x) = 0 \quad (6)$$

where $\eta(x)$ is a two-dimensional spinor $\eta(x) = (\eta^l(x), \eta^r(x))$. It is interesting that the two-component structure of Dirac spinor arises naturally when we consider fluctuations about exponentially localized wave functions. Here $g \equiv 2 \sinh(\gamma)$ is the velocity of the Dirac fermions while the mass of the Dirac fermions $m(x) = 2((\lambda V(x) + \cosh(\gamma)))$. Therefore, on a lattice, the Dirac fermions with disordered mass are the fluctuations of the nonrelativistic localized fermions. This would imply the absence of exponential localization for relativistic fermions. The defiance of localization by relativistic fermions has been the subject of various studies and our analysis provides a simple way to understand this intriguing result.

Next, we address the question whether the strong coupling fixed point which describes localized phase of Harper equation, the critical Ising model and the Dirac fermions with quasiperiodic disorder, implies universal multifractal exponents. We compute the $f(\alpha)$ curve (Fig. 2) describing the multifractal spectrum associated with the self-similar wave function or the inverse participation ratios \mathbf{P} , $\mathbf{P}(q, N) = \frac{\sum |\eta_n|^{2q}}{\sum |\eta_n|^2} \sim N^{-\tau(q)}$, $\alpha = \frac{d\tau}{dq}$ and $f(\alpha) = \alpha q - \tau(q)$ The free energy function $\tau(q)$ and its Legendre transform $f(\alpha)$ were found to be λ independent only for *positive values of q* and hence only left half of the $f(\alpha)$ curve is universal. Therefore, for quasiperiodic spin chains at the onset to LRO, scaling exponents for *only positive moments* of the participation ratio are universal. This can be viewed as a generalization of the universality statements of periodic spin chains to disordered spins.

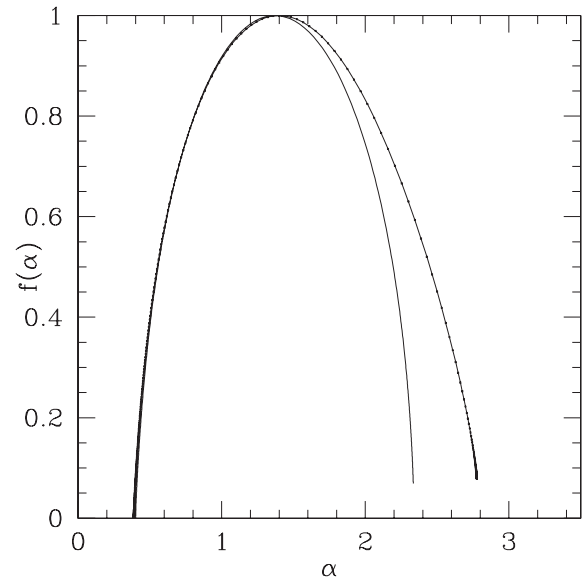


Fig. 2. Numerically obtained $f(\alpha)$ curves for $\lambda \rightarrow \infty$ (solid curve) and $\lambda = 1.5$ (lines with crosses).

Finally we investigate the role on correlations on the localization characteristics of massless spin excitations which obey Dirac equation. The fact that correlations would result in delocalization as originally shown by a random-dimer model [13] is an important result in localization theory [13]. For quasiperiodic disorder, dimer-type correlations can be introduced by replacing $\theta_n = 2\pi\sigma_g n$ in $V(\theta_n)$ by the iterates of the *supercritical* standard map, describing Hamiltonian systems with two degrees of freedom [14]

$$\theta_{n+1} + \theta_{n-1} - 2\theta_n = -\frac{K}{2\pi} \sin(2\pi\theta_n). \quad (7)$$

We use iterates that describe golden-mean cantor set (the remnant of the KAM torus beyond the onset to global stochasticity) which has been shown to exhibit dimer-type correlations, and leads to Bloch-type states for the non-relativistic fermions [14]. Here, we will confine to the Ising limit (can be obtained from (5) using $\gamma \rightarrow \infty$ and rescaling the parameters), described by,

$$\begin{aligned} \eta_{n+1}^1 + 2\lambda \cos(2\pi\theta_n)\eta_n^1 &= E\eta_n^2 \\ \eta_{n-1}^2 + 2\lambda \cos(2\pi\theta_n)\eta_n^2 &= E\eta_n^1. \end{aligned} \quad (8)$$

We determine the critical λ (the threshold for the onset to magnetic transition) as a function of K , the nonlinearity parameter of the two-dimensional map [2]. The localization characteristics of the massless mode of the Ising model are studied using an exact RG methodology [8]. In this approach, quasiperiodic models such as equation (7) with golden-mean incommensurability are decimated to a renormalized model defined only at the Fibonacci sites. Renormalization flow describing renormalized couplings at the Fibonacci sites provides an extremely accurate tool to distinguish extended, localized and critical states. Trivial fixed points of the RG describe extended states while critical states correspond to nontrivial fixed points. As shown

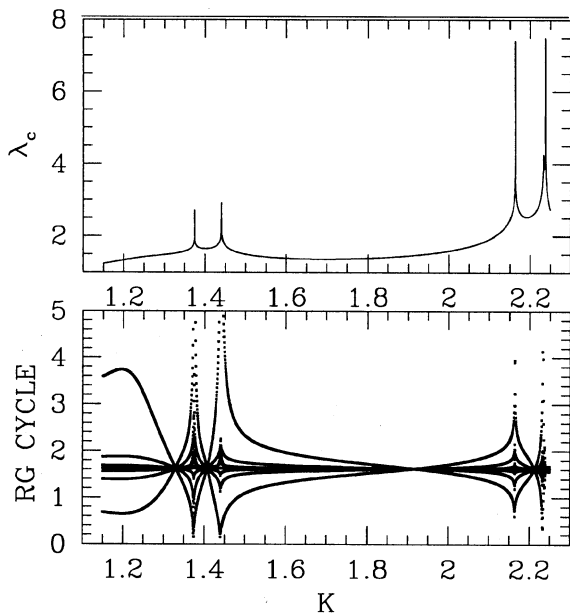


Fig. 3. (a) Critical λ_c as a function of K for the Ising model. (b) RG 6-cycle showing the variation in the renormalized coupling at Fibonacci iteration of the RG flow for $E = 0$.

in Figure 3, nontrivial 6-cycle (which also corresponds to six-cycle of the wave function η_n as shown in Fig. 1) degenerates to trivial fixed points at certain special parameter values. The origin of these trivial fixed points, has been traced to a *hidden dimer* in the quasiperiodic iterates describing the golden-cantorus [14]. At these points, the relativistic massless mode of the Ising model is ballistic. Therefore, relativistic fermions may become delocalized analogous to the nonrelativistic case due to dimer-type correlations.

Figure 3 shows an interesting interplay between the magnetic transition and the ballistic transition due to dimer-type correlations: the ballistic transitions where the relativistic mode is propagating seems to be sandwiched between two peaks corresponding to strong enhancement of (possibility divergent) strength of the inhomogeneous field needed for the onset to LRO. This phenomena again confirms the view that spin-fermion relationship may be an extremely useful means to understand the richness and complexity underlying a variety of new phenomena in disordered systems.

One-dimensional quantum spin chain in a transverse field at the onset to LRO describes two-dimensional layered Ising model. Therefore, our study relates universal aspects, described by Onsager universality, of two-dimensional Ising model to the universal aspects of the two-dimensional Bloch electron problem described by the Harper equation. This paper establishes a relationship between two important systems where geometry and integrability are of central importance. We believe that our results are valid for a variety of disorders including a large class of pseudorandom as well as random cases.

Finally, it should be noted that spin-fermion correspondence is also valid for time-dependent models. Kicked Harper model that has been extensively studied in quantum chaos literature [15,16] also describes XY spin chain in a periodically kicked inhomogeneous magnetic field [17]. By exploiting the spin-fermion correspondence, various new results in the kicked Harper model can provide a new way to understand many surprising results [18]. One of the interesting results is the independence of the critical exponents with respect to the discommensuration parameter σ_g in the limit $\sigma_g \rightarrow 0$. This defines a new aspect of the usual universality statements for the spin systems at the onset to magnetic transition and hence broadens the concept of universality to include disorder as well as time dependence.

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